Solution to Homework Assignment No. 5

1. (a) From the cofactor formula, we can have $\det \boldsymbol{A} = 3$ and

$$\left(\boldsymbol{A}^{-1}\right)_{11} = \frac{\boldsymbol{C}_{11}}{\det \boldsymbol{A}} = \frac{\begin{vmatrix} 3 & 0 \\ 7 & 1 \end{vmatrix}}{3} = 1$$

$$\left(\boldsymbol{A}^{-1}\right)_{12} = \frac{\boldsymbol{C}_{21}}{\det \boldsymbol{A}} = \frac{-\begin{vmatrix} 2 & 0 \\ 7 & 1 \end{vmatrix}}{3} = -\frac{2}{3}$$

$$\left(\boldsymbol{A}^{-1}\right)_{13} = \frac{\boldsymbol{C}_{31}}{\det \boldsymbol{A}} = \frac{\begin{vmatrix} 2 & 0 \\ 3 & 0 \end{vmatrix}}{3} = 0$$

$$\left(\boldsymbol{A}^{-1}\right)_{21} = \frac{\boldsymbol{C}_{12}}{\det \boldsymbol{A}} = \frac{-\begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix}}{3} = 0$$

$$\left(\boldsymbol{A}^{-1}\right)_{22} = \frac{\boldsymbol{C}_{22}}{\det \boldsymbol{A}} = \frac{\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}}{3} = \frac{1}{3}$$

$$\left(\boldsymbol{A}^{-1}\right)_{23} = \frac{\boldsymbol{C}_{32}}{\det \boldsymbol{A}} = \frac{-\begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix}}{3} = 0$$

$$\left(\boldsymbol{A}^{-1}\right)_{31} = \frac{\boldsymbol{C}_{13}}{\det \boldsymbol{A}} = \frac{\begin{vmatrix} 0 & 3 \\ 0 & 7 \end{vmatrix}}{3} = 0$$

$$\left(\boldsymbol{A}^{-1}\right)_{32} = \frac{\boldsymbol{C}_{23}}{\det \boldsymbol{A}} = \frac{-\begin{vmatrix} 1 & 2 \\ 0 & 7 \end{vmatrix}}{3} = -\frac{7}{3}$$

$$\left(\boldsymbol{A}^{-1}\right)_{33} = \frac{\boldsymbol{C}_{33}}{\det \boldsymbol{A}} = \frac{\begin{vmatrix} 1 & 2 \\ 0 & 3 \end{vmatrix}}{3} = 1.$$

Therefore, we can obtain the inverse of $\boldsymbol{A}$ as

$$\boldsymbol{A}^{-1} = \frac{1}{3}\begin{bmatrix} 3 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & -7 & 3 \end{bmatrix}.$$

(b) Since the matrix $\boldsymbol{A}$ is symmetric, the inverse of $\boldsymbol{A}$ is also symmetric. Then from the cofactor formula, we can have $\det \boldsymbol{A} = 4$ and

$$\left(\boldsymbol{A}^{-1}\right)_{11} = \frac{\boldsymbol{C}_{11}}{\det \boldsymbol{A}} = \frac{\begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix}}{4} = \frac{3}{4}$$

$$\left(\boldsymbol{A}^{-1}\right)_{21} = \frac{\boldsymbol{C}_{12}}{\det \boldsymbol{A}} = \frac{-\begin{vmatrix} -1 & -1 \\ 0 & 2 \end{vmatrix}}{4} = \frac{1}{2}$$

$$\left(\boldsymbol{A}^{-1}\right)_{22} = \frac{\boldsymbol{C}_{22}}{\det \boldsymbol{A}} = \frac{\begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix}}{4} = 1$$

$$\left(\boldsymbol{A}^{-1}\right)_{31} = \frac{\boldsymbol{C}_{13}}{\det \boldsymbol{A}} = \frac{\begin{vmatrix} -1 & 2 \\ 0 & -1 \end{vmatrix}}{4} = \frac{1}{4}$$

$$\left(\boldsymbol{A}^{-1}\right)_{32} = \frac{\boldsymbol{C}_{23}}{\det \boldsymbol{A}} = \frac{-\begin{vmatrix} 2 & -1 \\ 0 & -1 \end{vmatrix}}{4} = \frac{1}{2}$$

$$\left(\boldsymbol{A}^{-1}\right)_{33} = \frac{\boldsymbol{C}_{33}}{\det \boldsymbol{A}} = \frac{\begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix}}{4} = \frac{3}{4}$$

Therefore, we can obtain the inverse of $\boldsymbol{A}$ as

$$\boldsymbol{A}^{-1} = \frac{1}{4}\begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix}.$$

2. Since the Hadamard matrix $\boldsymbol{H}$ has orthogonal rows, the box is a hypercube and the volume is the multiplication of the lengths of the row vectors. And we know that every row vector has equal length which is $\sqrt{1^2 + 1^2 + 1^2 + 1^2} = 2$. Therefore,

$$|\det \boldsymbol{H}| = 2^4 = 16.$$

3. We know that

$$\begin{aligned} & \det\left(\boldsymbol{A} - \lambda \boldsymbol{I}\right) \\ = & \begin{vmatrix} a_{11} - \lambda & a_{12} & & \dots & a_{1n} \\ a_{21} & a_{22} - \lambda & & \dots & a_{2n} \\ \vdots & & \ddots & & \vdots \\ \vdots & & & \ddots & \vdots \\ a_{n1} & a_{n2} & & \dots & a_{nn} - \lambda \end{vmatrix} \\ = & \left(\lambda_1 - \lambda\right)\left(\lambda_2 - \lambda\right)\dots\left(\lambda_n - \lambda\right). \end{aligned}$$

The only term in the big formula for $\det(\boldsymbol{A} - \lambda \boldsymbol{I})$ which contains the λ^{n-1} terms is $(a_{11} - \lambda)(a_{22} - \lambda)\dots(a_{nn} - \lambda)$. Hence, the coefficient of λ^{n-1} in $\det(\boldsymbol{A} - \lambda \boldsymbol{I})$ is

$$(-1)^{n-1}\left(a_{11} + a_{12} + \dots + a_{nn}\right) = (-1)^{n-1} \operatorname{trace}\left(\boldsymbol{A}\right).$$

On the other hand, the coefficient of λ^{n-1} in $(\lambda_1 - \lambda)(\lambda_2 - \lambda)\ldots(\lambda_n - \lambda)$ is

$$(-1)^{n-1}(\lambda_1 + \lambda_2 + \ldots + \lambda_n).$$

Therefore,

$$\lambda_1 + \lambda_2 + \ldots + \lambda_n = \operatorname{trace}(\boldsymbol{A}).$$

4. (a) Let $\boldsymbol{u}_k = \begin{bmatrix} G_{k+1} \\ G_k \end{bmatrix}$. The relation between $\boldsymbol{u}_{k+1} = \begin{bmatrix} G_{k+2} \\ G_{k+1} \end{bmatrix}$ and $\boldsymbol{u}_k = \begin{bmatrix} G_{k+1} \\ G_k \end{bmatrix}$ is given by

$$\boldsymbol{u}_{k+1} = \begin{bmatrix} G_{k+2} \\ G_{k+1} \end{bmatrix} = \begin{bmatrix} \frac{1}{2}G_{k+1} + \frac{1}{2}G_k \\ G_{k+1} \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} G_{k+1} \\ G_k \end{bmatrix} = \boldsymbol{A}\boldsymbol{u}_k.$$

Then we have $\boldsymbol{u}_k = \boldsymbol{A}\boldsymbol{u}_{k-1} = \boldsymbol{A}\boldsymbol{A}\boldsymbol{u}_{k-2} = \boldsymbol{A}^2\boldsymbol{u}_{k-2} = \boldsymbol{A}^k\boldsymbol{u}_0$. To find $\boldsymbol{A}^k$, we first find the eigenvalues of $\boldsymbol{A}$.

$$\begin{aligned} \det(\boldsymbol{A} - \lambda\boldsymbol{I}) &= \begin{vmatrix} \frac{1}{2} - \lambda & \frac{1}{2} \\ 1 & -\lambda \end{vmatrix} \\ &= \lambda^2 - \frac{1}{2}\lambda - \frac{1}{2} \\ &= (\lambda - 1)\left(\lambda + \frac{1}{2}\right) = 0 \end{aligned}$$

$$\Longrightarrow \lambda = 1, -1/2.$$

For $\lambda_1 = 1$,

$$\boldsymbol{A} - \lambda_1\boldsymbol{I} = \begin{bmatrix} -1/2 & 1/2 \\ 1 & -1 \end{bmatrix}$$

and the corresponding eigenvector is

$$\boldsymbol{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

For $\lambda_2 = -1/2$,

$$\boldsymbol{A} - \lambda_2\boldsymbol{I} = \begin{bmatrix} 1 & 1/2 \\ 1 & 1/2 \end{bmatrix}$$

and the corresponding eigenvector is

$$\boldsymbol{x}_2 = \begin{bmatrix} -1/2 \\ 1 \end{bmatrix}.$$

Therefore, we have

$$\boldsymbol{A} = \boldsymbol{S}\Lambda\boldsymbol{S}^{-1} = \begin{bmatrix} 1 & -1/2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1/2 \end{bmatrix}\begin{bmatrix} 1 & -1/2 \\ 1 & 1 \end{bmatrix}^{-1}.$$

Then we write $\boldsymbol{u}_0$ as a linear combination of $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ as follows:

$$\boldsymbol{u}_0 = \begin{bmatrix} G_1 \\ G_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & -1/2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$\Longrightarrow \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 2/3 \\ -2/3 \end{bmatrix}$$

$$\Longrightarrow \boldsymbol{u}_0 = \frac{2}{3}\boldsymbol{x}_1 - \frac{2}{3}\boldsymbol{x}_2.$$

Then we can obtain

$$\begin{aligned} \boldsymbol{u}_k &= \boldsymbol{A}^k \boldsymbol{u}_0 \\ &= \boldsymbol{A}^k \left(\frac{2}{3}\boldsymbol{x}_1 - \frac{2}{3}\boldsymbol{x}_2 \right) \\ &= \frac{2}{3} \left(1^k \boldsymbol{x}_1 - \left(-\frac{1}{2} \right)^k \boldsymbol{x}_2 \right) \\ &= \frac{2}{3} \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} - \left(\frac{-1}{2} \right)^k \begin{bmatrix} -1/2 \\ 1 \end{bmatrix} \right) \\ &= \begin{bmatrix} G_{k+1} \\ G_k \end{bmatrix}. \end{aligned}$$

Therefore, we can have

$$G_k = \frac{2}{3} - \frac{2}{3}\left(-\frac{1}{2} \right)^k$$

for $k \geq 0$.

(b) When k goes to infinity, the term $(-1/2)^k$ goes to zero. Therefore, we can obtain

$$\lim_{k\to\infty} G_k = \lim_{k\to\infty} \left(\frac{2}{3} - \frac{2}{3}\left(-\frac{1}{2} \right)^k \right) = \frac{2}{3}.$$

5. (a) To diagonalize the matrix $\boldsymbol{A}$, we first find the eigenvalues of $\boldsymbol{A}$:

$$\begin{aligned} \det(\boldsymbol{A} - \lambda \boldsymbol{I}) &= \begin{vmatrix} 2-\lambda & -1 \\ -1 & 2-\lambda \end{vmatrix} \\ &= (2-\lambda)^2 - 1 \\ &= (\lambda - 1)(\lambda - 3) = 0. \end{aligned}$$

Then we can obtain $\lambda = 1, 3$. For $\lambda_1 = 1$,

$$\boldsymbol{A} - \lambda_1 \boldsymbol{I} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

and the corresponding eigenvector is

$$\boldsymbol{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

For $\lambda_2 = 3$,

$$\boldsymbol{A} - \lambda_2 \boldsymbol{I} = \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}$$

and the corresponding eigenvector is

$$\boldsymbol{x}_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Then we can have

$$\boldsymbol{S} = [\boldsymbol{x}_1 \ \boldsymbol{x}_2] = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

and the inverse of $\boldsymbol{S}$ given by

$$\boldsymbol{S}^{-1} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$

Therefore, the matrix $\boldsymbol{A}$ can be diagonalized as

$$\boldsymbol{A} = \boldsymbol{S\Lambda S}^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix}.$$

(b) We now have

$$\begin{aligned} \boldsymbol{A}^k &= \boldsymbol{S}\Lambda^k \boldsymbol{S}^{-1} \\ &= \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}^k \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3^k \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 3^k & -3^k \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} 1+3^k & 1-3^k \\ 1-3^k & 1+3^k \end{bmatrix}. \end{aligned}$$

6. To find an orthogonal matrix $\boldsymbol{Q}$, we first find the eigenvalues of the matrix

$$\boldsymbol{A} = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix}.$$

$$\begin{aligned} \det(\boldsymbol{A} - \lambda \boldsymbol{I}) &= \begin{vmatrix} 2-\lambda & 2 & 2 \\ 2 & -\lambda & 0 \\ 2 & 0 & -\lambda \end{vmatrix} \\ &= \lambda^2 (2-\lambda) + 4\lambda + 4\lambda \\ &= -\lambda^3 + 2\lambda^2 + 8\lambda \\ &= -\lambda(\lambda - 4)(\lambda + 2) = 0. \end{aligned}$$

Therefore, we have $\lambda = 4, -2, 0$. For $\lambda_1 = 4$, we have

$$\boldsymbol{A} - \lambda_1 \boldsymbol{I} = \begin{bmatrix} -2 & 2 & 2 \\ 2 & -4 & 0 \\ 2 & 0 & -4 \end{bmatrix}.$$

Then we can obtain the unit eigenvector

$$\boldsymbol{x}_1 = \frac{1}{\sqrt{6}} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}.$$

Similarly, for $\lambda_2 = -2$, we have

$$\boldsymbol{A} - \lambda_2 \boldsymbol{I} = \begin{bmatrix} 4 & 2 & 2 \\ 2 & 2 & 0 \\ 2 & 0 & 2 \end{bmatrix}$$

and the corresponding unit eigenvector

$$\boldsymbol{x}_2 = \frac{1}{\sqrt{3}} \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}.$$

For $\lambda_3 = 0$, we have

$$\boldsymbol{A} - \lambda_3 \boldsymbol{I} = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix}$$

and the corresponding unit eigenvector

$$\boldsymbol{x}_3 = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}.$$

We can check the orthogonality between eigenvectors:

$$\boldsymbol{x}_1^T \boldsymbol{x}_2 = \frac{1}{3\sqrt{2}} \begin{bmatrix} 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} = 0$$

$$\boldsymbol{x}_2^T \boldsymbol{x}_3 = \frac{1}{\sqrt{6}} \begin{bmatrix} -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} = 0$$

$$\boldsymbol{x}_1^T \boldsymbol{x}_3 = \frac{1}{2\sqrt{3}} \begin{bmatrix} 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} = 0.$$

Therefore, we can obtain an orthogonal matrix given by

$$\boldsymbol{Q} = \frac{1}{\sqrt{6}} \begin{bmatrix} 2 & -\sqrt{2} & 0 \\ 1 & \sqrt{2} & \sqrt{3} \\ 1 & \sqrt{2} & -\sqrt{3} \end{bmatrix}.$$

7. (a) Suppose $\boldsymbol{A}\boldsymbol{x} = \lambda \boldsymbol{x}$. Then we can take the complex conjugate on both sides and obtain

$$\overline{\boldsymbol{A}\boldsymbol{x}} = \overline{\lambda \boldsymbol{x}} \implies \boldsymbol{A}\overline{\boldsymbol{x}} = \overline{\lambda}\overline{\boldsymbol{x}}.$$

Since $\boldsymbol{A}$ is real, we have $\overline{\boldsymbol{A}} = \boldsymbol{A}$. Then we have the following relations:

$$\begin{aligned} & \boldsymbol{A}\overline{\boldsymbol{x}} = \overline{\lambda}\overline{\boldsymbol{x}} \\ \Longrightarrow\ & \overline{\boldsymbol{x}}^T\boldsymbol{A}^T = \overline{\lambda}\overline{\boldsymbol{x}}^T \\ \Longrightarrow\ & \overline{\boldsymbol{x}}^T\boldsymbol{A} = -\overline{\lambda}\overline{\boldsymbol{x}}^T. \end{aligned}$$

The last equation is true since $\boldsymbol{A}$ is skew-symmetric. Consider $\overline{\boldsymbol{x}}^T\boldsymbol{A}\boldsymbol{x}$, and we have

$$\overline{\boldsymbol{x}}^T(\boldsymbol{A}\boldsymbol{x}) = \overline{\boldsymbol{x}}^T(\lambda\boldsymbol{x}) = \lambda\overline{\boldsymbol{x}}^T\boldsymbol{x} = \lambda\|\boldsymbol{x}\|^2$$

and

$$\left(\overline{\boldsymbol{x}}^T\boldsymbol{A}\right)\boldsymbol{x} = \left(-\overline{\lambda}\overline{\boldsymbol{x}}^T\right)\boldsymbol{x} = -\overline{\lambda}\left(\overline{\boldsymbol{x}}^T\boldsymbol{x}\right) = -\overline{\lambda}\|\boldsymbol{x}\|^2.$$

Hence, we can have

$$\lambda = -\overline{\lambda}.$$

Therefore, a real skew-symmetric matrix has pure imaginary eigenvalues.

(b) Suppose λ is any eigenvalue of $\boldsymbol{A}$ and $\boldsymbol{x}$ is a corresponding unit eigenvector. Then we have

$$\boldsymbol{A}\boldsymbol{x} = \lambda\boldsymbol{x}.$$

It follows that

$$\|\boldsymbol{A}\boldsymbol{x}\|^2 = \|\lambda\boldsymbol{x}\|^2 = |\lambda|^2\|\boldsymbol{x}\|^2 = |\lambda|^2.$$

Also,

$$\|\boldsymbol{A}\boldsymbol{x}\|^2 = \left(\overline{\boldsymbol{A}\boldsymbol{x}}\right)^T(\boldsymbol{A}\boldsymbol{x}) = \overline{\boldsymbol{x}}^T\overline{\boldsymbol{A}}^T\boldsymbol{A}\boldsymbol{x} = \overline{\boldsymbol{x}}^T\boldsymbol{A}^T\boldsymbol{A}\boldsymbol{x} = \overline{\boldsymbol{x}}^T\boldsymbol{I}\boldsymbol{x} = \|\boldsymbol{x}\|^2 = 1$$

since $\boldsymbol{A}$ is an orthogonal matrix and $\boldsymbol{A}^T\boldsymbol{A} = \boldsymbol{I}$. Then we can have $|\lambda|^2 = 1$, yielding

$$|\lambda| = 1.$$

(c) Since $\boldsymbol{A}$ is a real skew-symmetric matrix and $\boldsymbol{A}^T\boldsymbol{A} = \boldsymbol{I}$, we know that $\boldsymbol{A}$ has all pure imaginary eigenvalues with $|\lambda| = 1$ from parts (a) and (b). Also, observe that the trace of $\boldsymbol{A}$ is zero. From Problem 3, we know that the sum of all eigenvalues of $\boldsymbol{A}$ is zero. Therefore, the four eigenvalues of $\boldsymbol{A}$ are $i, i, -i, -i$.

8. (a) We have

$$\begin{aligned} \boldsymbol{x}^T\boldsymbol{A}\boldsymbol{x} &= 2\left(x_1^2 + x_2^2 + x_3^2 - x_1x_2 - x_2x_3\right) \\ &= 2x_1^2 + 2x_2^2 + 2x_3^2 - x_1x_2 - x_2x_1 - x_2x_3 - x_3x_2. \end{aligned}$$

By inspection, we can obtain the symmetric matrix

$$\boldsymbol{A} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}.$$

To check whether $\boldsymbol{A}$ is positive definite, we compute the eigenvalues of $\boldsymbol{A}$.

$$\begin{aligned}\det(\boldsymbol{A}-\lambda\boldsymbol{I}) &= \begin{vmatrix} 2-\lambda & -1 & 0 \\ -1 & 2-\lambda & -1 \\ 0 & -1 & 2-\lambda \end{vmatrix} \\ &= (2-\lambda)\left(\lambda^2-4\lambda+2\right)=0 \\ \implies & \lambda = 2, 2+\sqrt{2}, 2-\sqrt{2}.\end{aligned}$$

Since all eigenvalues are positive, $\boldsymbol{A}$ is positive definite.

(b) We have

$$\begin{aligned}\boldsymbol{x}^T\boldsymbol{B}\boldsymbol{x} &= 2\left(x_1^2+x_2^2+x_3^2-x_1x_2-x_1x_3-x_2x_3\right) \\ &= 2x_1^2+2x_2^2+2x_3^2-x_1x_2-x_2x_1-x_1x_3-x_3x_1-x_2x_3-x_3x_2.\end{aligned}$$

By inspection, we can obtain the symmetric matrix

$$\boldsymbol{B} = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}.$$

To check whether $\boldsymbol{B}$ is semidefinite, we compute the eigenvalues of $\boldsymbol{B}$.

$$\begin{aligned}\det(\boldsymbol{B}-\lambda\boldsymbol{I}) &= \begin{vmatrix} 2-\lambda & -1 & -1 \\ -1 & 2-\lambda & -1 \\ -1 & -1 & 2-\lambda \end{vmatrix} \\ &= -\lambda(\lambda-3)^2=0 \\ \implies & \lambda = 3, 3, 0.\end{aligned}$$

Since all eigenvalues are nonnegative, $\boldsymbol{B}$ is positive semidefinite.

9. Let $\boldsymbol{A} = \begin{bmatrix} 1 & 1/2 \\ 1/2 & 1 \end{bmatrix}$. Then

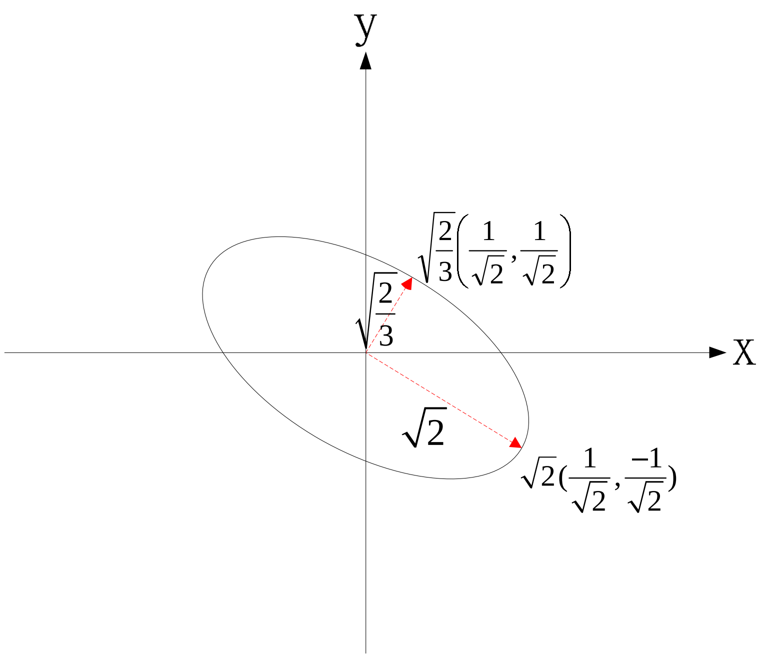
$$\boldsymbol{x}^T\boldsymbol{A}\boldsymbol{x} = x^2+xy+y^2$$

where $\boldsymbol{x} = \begin{bmatrix} x \\ y \end{bmatrix}$. By the spectral theorem,

$$\boldsymbol{A} = \boldsymbol{Q}\boldsymbol{\Lambda}\boldsymbol{Q}^T = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1/2 & 0 \\ 0 & 3/2 \end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}.$$

Then we have

$$\begin{aligned}\boldsymbol{x}^T\boldsymbol{A}\boldsymbol{x} &= \left(\boldsymbol{Q}^T\boldsymbol{x}\right)^T\boldsymbol{\Lambda}\left(\boldsymbol{Q}^T\boldsymbol{x}\right) \\ &= \begin{bmatrix} \frac{x-y}{\sqrt{2}} & \frac{x+y}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} 1/2 & 0 \\ 0 & 3/2 \end{bmatrix}\begin{bmatrix} \frac{x-y}{\sqrt{2}} \\ \frac{x+y}{\sqrt{2}} \end{bmatrix} \\ &= \frac{1}{2}\left(\frac{x-y}{\sqrt{2}}\right)^2+\frac{3}{2}\left(\frac{x+y}{\sqrt{2}}\right)^2 \\ &= \frac{1}{2}X^2+\frac{3}{2}Y^2\end{aligned}$$



where $X = (x - y)/\sqrt{2}$ and $Y = (x + y)/\sqrt{2}$. The equation can be rewritten as

$$\frac{X^2}{2} + \frac{Y^2}{(2/3)} = 1.$$

Then we can obtain the half-lengths of its axes are

$$\sqrt{2}, \sqrt{2/3}.$$

The tilted ellipse is drawn as above.

10. We can find the eigenvalues of each matrix as follows.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} : \ \lambda = 1, 1$$
$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} : \ \lambda = 1, -1$$
$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} : \ \lambda = 0, 1$$
$$\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} : \ \lambda = 0, 1$$
$$\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} : \ \lambda = 0, 1$$
$$\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} : \ \lambda = 0, 1.$$

Since all 2×2 matrices with eigenvalues 1 and 0 are similar to each other, the following matrices are similar:

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}.$$

The matrices

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

are similar to themselves.

11. (a) We have

$$\boldsymbol{A}^T\boldsymbol{A} = \begin{bmatrix} 10 & 20 \\ 20 & 40 \end{bmatrix}.$$

Then

$$0 = \det\left(\boldsymbol{A}^T\boldsymbol{A} - \lambda\boldsymbol{I}\right) = \begin{vmatrix} 10-\lambda & 20 \\ 20 & 40-\lambda \end{vmatrix} = \lambda(\lambda - 50) \Longrightarrow \lambda = 50, 0.$$

For $\lambda_1 = 50$, the corresponding unit eigenvector is $\boldsymbol{v}_1 = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

For $\lambda_2 = 0$, the corresponding unit eigenvector is $\boldsymbol{v}_2 = \frac{1}{\sqrt{5}}\begin{bmatrix} 2 \\ -1 \end{bmatrix}$.

(b) Since $\sigma_1 = \sqrt{\lambda_1}$, we have $\sigma_1 = 5\sqrt{2}$. Then we can find

$$\boldsymbol{u}_1 = \frac{\boldsymbol{A}\boldsymbol{v}_1}{\sigma_1} = \frac{\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}\frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ 2 \end{bmatrix}}{5\sqrt{2}} = \frac{1}{\sqrt{10}}\begin{bmatrix} 1 \\ 3 \end{bmatrix}.$$

Now we verify that $\boldsymbol{u}_1$ is a unit eigenvector of $\boldsymbol{A}\boldsymbol{A}^T$ as follows:

$$\left(\boldsymbol{A}\boldsymbol{A}^T\right)\boldsymbol{u}_1 = \begin{bmatrix} 5 & 15 \\ 15 & 45 \end{bmatrix}\frac{1}{\sqrt{10}}\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \frac{1}{\sqrt{10}}\begin{bmatrix} 50 \\ 150 \end{bmatrix} = 50\boldsymbol{u}_1$$

$$\|\boldsymbol{u}_1\|^2 = \boldsymbol{u}_1^T\boldsymbol{u}_1 = \frac{1}{10}\begin{bmatrix} 1 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \end{bmatrix} = 1.$$

(c) For $\lambda_2 = 0$, we can find a unit eigenvector for $\boldsymbol{A}\boldsymbol{A}^T$ as

$$\boldsymbol{u}_2 = \frac{1}{\sqrt{10}}\begin{bmatrix} 3 \\ -1 \end{bmatrix}.$$

Therefore, we have

$$\boldsymbol{A} = \boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^T$$

where

$$\boldsymbol{U} = \frac{1}{\sqrt{10}}\begin{bmatrix} 1 & 3 \\ 3 & -1 \end{bmatrix}$$

$$\boldsymbol{\Sigma} = \begin{bmatrix} 5\sqrt{2} & 0 \\ 0 & 0 \end{bmatrix}$$

$$\boldsymbol{V} = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}.$$

12. (a) We have

$$A^T A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}.$$

Let

$$v_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} \sin \pi/4 \\ \sin 2\pi/4 \\ \sin 3\pi/4 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/\sqrt{2} \\ 1/2 \end{bmatrix}$$

$$v_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} \sin 2\pi/4 \\ \sin 4\pi/4 \\ \sin 6\pi/4 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix}$$

$$v_3 = \frac{1}{\sqrt{2}} \begin{bmatrix} \sin 3\pi/4 \\ \sin 6\pi/4 \\ \sin 9\pi/4 \end{bmatrix} = \begin{bmatrix} 1/2 \\ -1/\sqrt{2} \\ 1/2 \end{bmatrix}.$$

Then we can have

$$A^T A v_1 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1/2 \\ 1/\sqrt{2} \\ 1/2 \end{bmatrix} = \begin{bmatrix} 1 - \frac{1}{\sqrt{2}} \\ -1 + \sqrt{2} \\ 1 - \frac{1}{\sqrt{2}} \end{bmatrix} = \left(2 - \sqrt{2}\right) \begin{bmatrix} 1/2 \\ 1/\sqrt{2} \\ 1/2 \end{bmatrix} = \lambda_1 v_1$$

$$A^T A v_2 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} \sqrt{2} \\ 0 \\ -\sqrt{2} \end{bmatrix} = 2 \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix} = \lambda_2 v_2$$

$$A^T A v_3 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1/2 \\ -1/\sqrt{2} \\ 1/2 \end{bmatrix} = \begin{bmatrix} 1 + \frac{1}{\sqrt{2}} \\ -1 - \sqrt{2} \\ 1 + \frac{1}{\sqrt{2}} \end{bmatrix} = \left(2 + \sqrt{2}\right) \begin{bmatrix} 1/2 \\ -1/\sqrt{2} \\ 1/2 \end{bmatrix} = \lambda_3 v_3.$$

Therefore, the columns of V have $A^T A v = \lambda v$ with $\lambda = 2 - \sqrt{2}, 2, 2 + \sqrt{2}$.

(b) We can have

$$\begin{aligned} AV &= \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{\sqrt{2}} & \frac{1}{2} \\ \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \\ \frac{1}{2} & \frac{-1}{\sqrt{2}} & \frac{1}{2} \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{2} & \frac{1}{\sqrt{2}} & \frac{1}{2} \\ -\frac{1}{2} + \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & -\frac{1}{2} - \frac{1}{\sqrt{2}} \\ \frac{1}{2} - \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & \frac{1}{2} + \frac{1}{\sqrt{2}} \\ -\frac{1}{2} & \frac{1}{\sqrt{2}} & -\frac{1}{2} \end{bmatrix}. \end{aligned}$$

Let $\boldsymbol{AV} = [\boldsymbol{x}_1\ \boldsymbol{x}_2\ \boldsymbol{x}_3]$. Then we have

$$\boldsymbol{x}_1^T\boldsymbol{x}_2 = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2}+\frac{1}{\sqrt{2}} & \frac{1}{2}-\frac{1}{\sqrt{2}} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = 0$$

$$\boldsymbol{x}_2^T\boldsymbol{x}_3 = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2}-\frac{1}{\sqrt{2}} \\ \frac{1}{2}+\frac{1}{\sqrt{2}} \\ -\frac{1}{2} \end{bmatrix} = 0$$

$$\boldsymbol{x}_1^T\boldsymbol{x}_3 = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2}+\frac{1}{\sqrt{2}} & \frac{1}{2}-\frac{1}{\sqrt{2}} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2}-\frac{1}{\sqrt{2}} \\ \frac{1}{2}+\frac{1}{\sqrt{2}} \\ -\frac{1}{2} \end{bmatrix} = 0.$$

Therefore, the columns of $\boldsymbol{AV}$ are orthogonal.

(c) We have

$$\boldsymbol{A}^T = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}.$$

Perform Gaussian elimination as follows:

$$\begin{aligned} & \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix} \\ \Longrightarrow & \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix} \\ \Longrightarrow & \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix}. \end{aligned}$$

Then we can obtain

$$\boldsymbol{u}_4 = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

(d) From parts (b) and (c), we have the vectors

$$\boldsymbol{u}_1 = \frac{\boldsymbol{x}_1}{\sigma_1} = \frac{\boldsymbol{x}_1}{\sqrt{\lambda_1}} = \frac{1}{\sqrt{2-\sqrt{2}}}\begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2}+\frac{1}{\sqrt{2}} \\ \frac{1}{2}-\frac{1}{\sqrt{2}} \\ -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{8-4\sqrt{2}}} \\ \sqrt{\frac{2-\sqrt{2}}{8}} \\ -\sqrt{\frac{2-\sqrt{2}}{8}} \\ -\frac{1}{\sqrt{8-4\sqrt{2}}} \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{4+2\sqrt{2}}}{4} \\ \frac{\sqrt{4-2\sqrt{2}}}{4} \\ -\frac{\sqrt{4-2\sqrt{2}}}{4} \\ -\frac{\sqrt{4+2\sqrt{2}}}{4} \end{bmatrix}$$

$$\boldsymbol{u}_2 = \frac{\boldsymbol{x}_2}{\sigma_2} = \frac{\boldsymbol{x}_2}{\sqrt{\lambda_2}} = \frac{1}{\sqrt{2}}\begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$$

$$\boldsymbol{u}_3 = \frac{\boldsymbol{x}_3}{\sigma_3} = \frac{\boldsymbol{x}_3}{\sqrt{\lambda_3}} = \frac{1}{\sqrt{2+\sqrt{2}}}\begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2}-\frac{1}{\sqrt{2}} \\ \frac{1}{2}+\frac{1}{\sqrt{2}} \\ -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{8+4\sqrt{2}}} \\ -\sqrt{\frac{2+\sqrt{2}}{8}} \\ \sqrt{\frac{2+\sqrt{2}}{8}} \\ -\frac{1}{\sqrt{8+4\sqrt{2}}} \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{4-2\sqrt{2}}}{4} \\ -\frac{\sqrt{4+2\sqrt{2}}}{4} \\ \frac{\sqrt{4+2\sqrt{2}}}{4} \\ -\frac{\sqrt{4-2\sqrt{2}}}{4} \end{bmatrix}$$

$$\boldsymbol{u}_4 = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

Therefore, we have $\boldsymbol{A} = \boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^T$, where

$$\boldsymbol{U} = \begin{bmatrix} \frac{\sqrt{4+2\sqrt{2}}}{4} & \frac{1}{2} & \frac{\sqrt{4-2\sqrt{2}}}{4} & \frac{1}{2} \\ \frac{\sqrt{4-2\sqrt{2}}}{4} & -\frac{1}{2} & -\frac{\sqrt{4+2\sqrt{2}}}{4} & \frac{1}{2} \\ -\frac{\sqrt{4-2\sqrt{2}}}{4} & -\frac{1}{2} & \frac{\sqrt{4+2\sqrt{2}}}{4} & \frac{1}{2} \\ -\frac{\sqrt{4+2\sqrt{2}}}{4} & \frac{1}{2} & -\frac{\sqrt{4-2\sqrt{2}}}{4} & \frac{1}{2} \end{bmatrix}$$

$$\boldsymbol{\Sigma} = \begin{bmatrix} \sqrt{2-\sqrt{2}} & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2+\sqrt{2}} \\ 0 & 0 & 0 \end{bmatrix}$$

$$\boldsymbol{V} = \begin{bmatrix} \frac{1}{2} & \frac{1}{\sqrt{2}} & \frac{1}{2} \\ \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \\ \frac{1}{2} & \frac{-1}{\sqrt{2}} & \frac{1}{2} \end{bmatrix}.$$